EQUATIONS OF MOTION OF A THIN LAYER OF FLUID ON THE SURFACE OF A ROTATING BODY OF REVOLUTION

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P. L. Kapitsa [1, 2] and his followers have considered the problem of the flow of a thin film of viscous fluid on a vertical wall. O. V. Golubeva [3] has generalized the equations of two-dimensional flows of an ideal fluid to include the motion in a film covering a curved surface. The motion of a thin layer of fluid over a sphere is considered in the theory of tidal currents.

The present paper concerns itself with a study of the unsteady flow of a thin layer of fluid on the surface of a body of revolution rotating with variable angular velocity about its axis and subjected to an external axial body force in the presence of precipitation of fluid particles onto the free surface. The system of differential equations of this motion is obtained from the general equations of motion of a viscous fluid in a curvilinear coordinate system that moves with the body. The system of equations thus obtained is simplified by averaging the velocity components over the layer thickness.

NOTATION

$R(x)$ is the distance from points on the surface of the body of revolution to the axis; $j(t)$ is the acceleration of the external body force, directed along the axis of revolution; q is the volume of precipitation of fluid particles per unit time per unit surface area, $q = q(x, \varphi, t)$; u_i are velocity components at the axis $(i = 1, 2, 3)$; ω is the angular velocity; ρ is the density; τ_{zx} and $\tau_{z\varphi}$ are tangential stress components; μ is the viscosity; p is the pressure; h is the depth of the flow; v_i are components of velocity averaged over the depth at the axis $(i = 1, 2)$.

1. Let us consider the unsteady flow of a thin layer of viscous fluid on the surface of a body of revolution which revolves about its own axis and moves progressively along it. The absolute magnitudes of the angular velocity ω (t) and of the rate of translational motion may vary with time.

The motion of the fluid layer is considered in an orthogonal curvilinear coordinate system x, φ, z, moving with the body; here the x-coordinate is measured along the meridian arc of the surface of the body of revolution, and the z-coordinate is measured along a normal to the surface of the body directed inward in relation to the fluid layer (see figure).

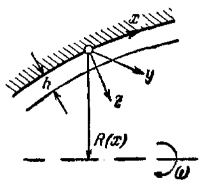

Fig. 1

Let the contour of the body of revolution be specified by the radius $R(x)$, which is large compared with the layer thickness. It is also assumed that the contour is sufficiently smooth or that the radius of curvature of the meridian section is large in comparison with the layer thickness. If the layer thickness varies smoothly over the surface, then the curvature of the free surface will also be relatively small. In this case it is natural to disregard surface tension.

In addition to forces due to the flow of the fluid in the rotating coordinate system, the fluid is also acted upon by an external body force directed along the axis of rotation with acceleration $j(t)$, where t is time. This force may be associated with the accelerated motion of the body along the axis, and in a particular case can be represented by the acceleration of gravity g (vertical axis).

Particles of the same fluid may be precipitated onto the free surface of the layer at an average rate $q(x, \varphi, t)$, which represents the volume of particles precipitated per unit time per unit surface area. The pressure of the gas medium at the free surface of the flow, p_0, will be regarded as constant.

Assuming that the layer thickness is small in comparison with the radii of curvature of the free surface and the surface of the body, from the general equations of motion of a viscous fluid in the selected coordinate system we get the following system of approximate equations:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{u_2}{R}\frac{\partial u_1}{\partial \varphi} + u_3 \frac{\partial u_1}{\partial z} - \frac{u_2^2}{R} R' = j\sqrt{1 - R'^2} + \omega^2 R R' + 2\omega u_2 R' - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}, \qquad (1.1)$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + \frac{u_2}{R}\frac{\partial u_2}{\partial \varphi} + u_3 \frac{\partial u_2}{\partial z} + \frac{u_1 u_2}{R} R' = -\frac{\partial \omega}{\partial t} R - 2\omega R' u_1 - \frac{1}{\rho R}\frac{\partial p}{\partial \varphi} + \frac{1}{\rho}\frac{\partial \tau_{z\varphi}}{\partial z}, \qquad (1.2)$$

$$\frac{u_2^2}{R}\sqrt{1 - R'^2} = jR' - \omega^2 R\sqrt{1 - R'^2} - 2\omega u_2 \sqrt{1 - R'^2} - \frac{1}{\rho}\frac{\partial p}{\partial z}, \qquad (1.3)$$

$$\frac{\partial (u_1 R)}{\partial x} + \frac{\partial u_2}{\partial \varphi} + \frac{\partial (u_3 R)}{\partial z} = 0. \qquad (1.4)$$

Here u_1, u_2, and u_3 are the velocity components with respect to the x, y, and z coordinate axes, p is the pressure, ρ is the density τ_{zx} and $\tau_{z\varphi}$ are tangential stresses. In the case of laminar flow

$$\tau_{zx} = \mu \frac{\partial u_1}{\partial z}, \qquad \tau_{z\varphi} = \mu \frac{\partial u_2}{\partial z}. \qquad (1.5)$$

Here, when $z = 0$ we have

$$u_1 = u_2 = u_3 = 0. \qquad (1.6)$$

In the case of precipitation of particles of the same fluid onto the free surface ("rain") the dynamic effect of this phenomenon can be approximately represented by means of the momentum theorem in the form of averaged tangential stresses at the free surface

$$\tau_{zx} = \rho q (u_1' - u_1), \qquad \tau_{z\varphi} = \rho q (u_2' - u_2), \qquad \text{for } z = h. \qquad (1.7)$$

Here u_1' and u_2' are the velocity components of the joining particles with respect to the x, φ coordinates before joining (defined with respect to the coordinate system moving with the body). If there is no particle fall-out, then the tangential stresses at the free surface are assumed to vanish

$$\tau_{zx} = 0, \qquad \tau_{z\varphi} = 0, \qquad \text{for } z = h. \qquad (1.8)$$

Note that in addition to Eqs. (1.7) or (1.8) and the condition that $p(x, \varphi, h) = p_0 = \text{const}$, the following condition should be satisfied at the free surface $z = h(x, \varphi, t)$,

$$\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x} + \frac{u_2}{R}\frac{\partial h}{\partial \varphi} = u_3 + q. \qquad (1.9)$$

In addition to the aforementioned boundary conditions at the free surface and at the surface of the body, the layer thickness h and the velocity distribution over its thickness must be specified in at least one cross section (for example, the initial cross section $x = 0$). In considering problems of unsteady flow, the above boundary conditions must be supplemented by initial conditions expressing the distribution of layer thickness and velocity at the initial time instant.

The problem as stated above can be simplified by using integral principles, in the same way as in boundary layer theory. It turns out that for a a thin flow with a free surface the use of these principles results in a formulation of the problem which is similar to shallow water theory or to the hydraulic theory of flow in open channels.

2. To simplify the differential equations obtained, we now average the velocity components over the layer thickness and disregard the nonuniformity of velocity distribution over the thickness. We denote

$$v_i = \frac{1}{h}\int_0^h u_i\,dz. \tag{2.1}$$

Disregarding the nonuniformity of the velocity distribution over the layer thickness, we assume that

$$\int_0^h u_i u_k\,dz \approx v_i v_k h, \qquad \int_z^h u_i u_k\,dz \approx v_i v_k (h - z). \tag{2.2}$$

Integrating the continuity equation (1.4) with respect to z from 0 to h, and using Eq. (1.9), we get

$$\frac{\partial h}{\partial t} + \frac{\partial (v_1 h)}{\partial x} + \frac{R'}{R} v_1 h + \frac{1}{R}\frac{\partial (v_2 h)}{\partial \varphi} = q. \tag{2.3}$$

Integrating in a similar manner the third dynamic equation, making use of (2.2), we obtain an approximate expression for the pressure distribution over the layer thickness

$$p = p_0 + \rho f (h - z), \quad f = \left(\omega^2 R + 2\omega v_2 + \frac{v_2^2}{R}\right)\sqrt{1 - R'^2} - jR'. \tag{2.4}$$

Hence

$$\frac{\partial p}{\partial x} = \rho f \frac{\partial h}{\partial x} + \rho (h - z)\frac{\partial f}{\partial x}, \quad \frac{\partial p}{\partial \varphi} = \rho f \frac{\partial h}{\partial \varphi} + \rho (h - z)\frac{\partial f}{\partial \varphi}. \tag{2.5}$$

Consideration of (2.5) shows that, by virtue of the assumed smallness of h/R and of the curvature of the meridian cross section of the body surface, which is characterized by R"(x), the second terms in these expressions are small in comparison with the first. Hence we may consider that

$$\frac{\partial p}{\partial x} = \rho f \frac{\partial h}{\partial x}, \qquad \frac{\partial p}{\partial \varphi} = \rho f \frac{\partial h}{\partial \varphi}. \tag{2.6}$$

Substituting these expressions into (1.1) and (1.2), integrating the latter over the thickness of the layer, and making use of (2.2), we obtain the approximate equations

$$\frac{\partial (v_1 h)}{\partial t} + \frac{\partial (v_1^2 h)}{\partial x} + \frac{1}{R}\frac{\partial (v_1 v_2 h)}{\partial \varphi} + \frac{R'}{R}(v_1^2 - v_2^2) h - q v_1 = F_1 h, \tag{2.7}$$

$$F_1 = j\sqrt{1 - R'^2} + \omega^2 R R' + 2\omega R' v_2 - f\frac{\partial h}{\partial x} + \frac{1}{\rho h}\left[\tau_{zx}(x, \varphi, h) - \tau_{zx}(x, \varphi, 0)\right],$$

$$\frac{\partial (v_2 h)}{\partial t} + \frac{\partial (v_1 v_2 h)}{\partial x} + \frac{1}{R}\frac{\partial (v_2^2 h)}{\partial \varphi} + 2\frac{R'}{R} v_1 v_2 h - q v_2 = F_2 h, \tag{2.8}$$

$$F_2 = -\left(\frac{d\omega}{dt} R + 2\omega R' v_1\right) - \frac{1}{R} f \frac{\partial h}{\partial \varphi} + \frac{1}{\rho h}\Big[\left[\tau_{z\varphi}(x, \varphi, h) - \tau_{z\varphi}(x, \varphi, 0)\right]\Big].$$

Using Eq. (2.3), we can also represent these equations in the form

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + \frac{v_2}{R}\frac{\partial v_1}{\partial \varphi} - R'\frac{v_2^2}{R} = F_1 \tag{2.9}$$

$$\frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x} + \frac{v_2}{R}\frac{\partial v_2}{\partial \varphi} + R'\frac{v_1 v_2}{R} = F_2 \tag{2.10}$$

According to (1.7)

$$\tau_{zx}(x, \varphi, h) = \rho q (u_1' - v_1), \qquad \tau_{z\varphi}(x, \varphi, h) = \rho q (u_2' - v_2). \tag{2.11}$$

In the same way as in boundary layer theory, the tangential stresses at the surface of the body can be approximately related to the average velocities by equations of the form

$$\tau_{zx}(x, \varphi, 0) = {}^1/_8 \lambda \rho v v_1, \qquad \tau_{z\varphi}(x, \varphi, 0) = {}^1/_8 \lambda \rho v v_2. \tag{2.12}$$

Here λ is a dimensionless resistance coefficient for uniform flow in pipes, and v is the absolute value of the velocity.

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